Announcements

1) Practice Problems for Quiz 2 up (Quiz 2 this Thursday)

2) HW 3 redo - solve problem #2 by hand, using variation of parameters, up to the point where you find the form of $u'(t), v'(t)$. Use your favorite resource to solve for u, v, find the particular solution. Due in one week!

3) Exam 2 Tuesday next week
(tentatively)

Example 1: Solve

$$t^2 y'' + 3t y' + y = 0.$$

We assume $y = t^r$ for some real number r.

$$y'(t) = r t^{r-1}, \quad y''(t) = r(r-1) t^{r-2}.$$

Substitute back into equation.

$$r(r-1) t^{r-2} (t^2) + 3t r t^{r-1} + t^r = 0,$$

so

$$t^r (r^2 - r + 3r + 1) = 0$$

Simplifying,

$$t^r(r^2+2r+1)=0.$$

This holds for all t only when

$$r^2+2r+1=0$$

$$(r+1)^2=0$$

$$\boxed{r=-1}$$

So $\boxed{y(t)=ct^{-1}}$

is one solution.

Since the equation is second degree, we expect there to be a second solution that is linearly independent from the first.

Can we find it?

Standard Form

Given a second order equation of the form

$$a(t)y'' + b(t)y' + c(t)y = f(t),$$

we may assume $a(t) \neq 0$ (at least "most" of the time).

We can then divide the equation by $a(t)$ to produce

$$\boxed{y'' + \frac{b(t)}{a(t)}y' + \frac{c(t)}{a(t)}y = \frac{f(t)}{a(t)}}$$

Setting $p(t) = \frac{b(t)}{a(t)}$, $q(t) = \frac{c(t)}{a(t)}$,

and $h(t) = \frac{f(t)}{a(t)}$, we

arrive at the Standard form

of the equation:

$$\boxed{y'' + p(t)y' + q(t)y = h(t)}$$

Let's put

$$t^2 y'' + 3t y' + y = 0$$

in standard form. Dividing by t^2, we get

$$\boxed{y'' + \frac{3}{t} y' + \frac{1}{t^2} y = 0}$$

So $p(t) = \dfrac{3}{t}$, $q(t) = \dfrac{1}{t^2}$

Another outrageous trick!

If y_1 is a nonzero solution, then

$$\boxed{y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2} dt}$$

is a second, linearly independent solution

Plug our functions in:

$$p(t) = \frac{3}{t}, \quad y_1(t) = \frac{1}{t}$$

Then

$$e^{-\int \frac{3}{t} dt}$$

$$= e^{-3\ln(t)}$$

$$= t^{-3}, \text{ and so}$$

$$\boxed{y_2(t) = \frac{1}{t} \int \frac{\left(\frac{1}{t^3}\right)}{\left(\frac{1}{t}\right)^2} dt}$$

So

$$y_2(t) = \frac{1}{t} \int \frac{1}{t}\,dt$$

$$= \frac{\ln(t)}{t}, \quad \text{so}$$

a general solution to

the equation is

$$\boxed{y(t) = C_1 \frac{1}{t} + C_2 \frac{\ln(t)}{t}}$$

Theorem: (solutions)

Suppose that $p(t)$, $q(t)$, and $h(t)$ are continuous on an interval (a,b) that contains the point t_0. Then there exists a unique solution $y(t)$ to the initial value problem

$$\boxed{\begin{gathered} y''(t)+p(t)y'(t)+q(t)y(t)=h(t), \\ y(t_0)=y_0,\ y(t_1)=y_1 \end{gathered}}$$

for any numbers y_0, y_1.

General Method

How to solve

$$y'' + p(t)y'(t) + q(t)y(t) = h(t)$$

1) Solve the homogeneous equation

$$y'' + p(t)y'(t) + q(t)y(t) = 0$$

by:

a) guessing a solution $y_1(t)$

b) finding a second, linearly independent solution $y_2(t)$

Use the formula

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{(y_1(t))^2} dt$$

General solutions are of the form

$$z(t) = C_1 y_1(t) + C_2 y_2(t)$$

b) Use variation of parameters to find a single solution $y_p(t)$ to the original equation:

Assume $y_p(t) = u(t)y_1(t) + v(t)y_2(t)$

and that

$$u'(t)y_1(t) + v'(t)y_2(t) = 0$$

$$u'(t)y_1'(t) + v'(t)y_2'(t) = h(t)$$

Solve for u and v (usually using a computer)

3) Solutions are of the form

$$\boxed{y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)}$$

Cauchy-Euler Equations

A second order differential equation of the form

$$\boxed{t^2 y'' + a t y' + b y = f(t)}$$

is called a Cauchy-Euler equation. In principle, we know how to solve these! We can always find a solution t^r for the homogeneous equation.